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q-DEFORMED ALGEBRAS $U_q(\mathrm{so}_n)$ AND THEIR REPRESENTATIONS

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Abstract

For the nonstandard q-deformed algebras $U_q(\mathrm{so}_n)$, defined recently in terms of trilinear relations for generating elements, most general finite dimensional irreducible representations directly corresponding to those of nondeformed algebras $\mathrm{so}(n)$ (i.e., characterized by the same sets of only integers or only half-integers as in highest weights of the latter) are given explicitly in a q-analogue of Gel'fand-Tsetlin basis. Detailed proof, for q not equal to a root of unity, that representation operators indeed satisfy relevant (trilinear) relations and define finite dimensional irreducible representations is presented. The results show perfect suitability of the Gel'fand-Tsetlin formalism concerning (nonstandard) q-deformation of $\mathrm{so}(n)$.

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1. Introduction

The so called *standard* quantum deformations $U_q(\mathrm{B}_r)$, $U_q(\mathrm{D}_r)$ of Lie algebras of the orthogonal groups $\mathrm{SO}(2r+1)$, $\mathrm{SO}(2r)$ were given by Drinfeld [1] and Jimbo [2]. However, there are some reasons why to consider a q-deformation of the orthogonal Lie algebras $\mathrm{so}(n,\mathbf{C})$ other than the standard quantum algebras $U_q(\mathrm{B}_r)$, $U_q(\mathrm{D}_r)$. In order to construct explicitly (possibly, all) finite dimensional irreducible representations of q-deformed algebras $U_q(\mathrm{so}(n,\mathbf{C}))$ and their compact real forms $U_q(\mathrm{so}_n)$, one needs a q-analogue of the Gel'fand–Tsetlin (GT) basis and the GT action formulas [3],[4] which requires existence of canonical embeddings q-analogous to the chain $\mathrm{so}(n,\mathbf{C}) \supset \mathrm{so}(n-1,\mathbf{C}) \supset \cdots \supset \mathrm{so}(3,\mathbf{C})$. Obviously, such a chain does not exist for the standard quantum algebras $U_q(\mathrm{B}_r)$ and $U_q(\mathrm{D}_r)$, only single-type embedding $U_q(\mathrm{B}_{r-1}) \subset U_q(\mathrm{B}_r)$ and embedding $U_q(\mathrm{D}_{r-1}) \subset U_q(\mathrm{D}_r)$ do hold. Moreover, as it was pointed out by Faddeev, Reshetikhin and Takhtadzyan [5], standard quantum orthogonal groups $\mathrm{SO}_q(n,\mathbf{C})$ defined on the base of R-matrix do not possess, when $n > 3$, real forms of Lorentz signature (which are necessary for investigating important aspects of quantized n-dimensional space-times). Again, in developing representation theory of a q-analogue of the Lorentz algebras $\mathrm{so}(n,1)$ in a way parallel to the classical ($q=1$) case, see [6] and references therein, one also needs q-analogues of the GT basis and the GT formulas which certainly require that the chain of inclusions $U_q(\mathrm{so}(n,1)) \supset U_q(\mathrm{so}(n)) \supset U_q(\mathrm{so}(n-1)) \supset \cdots \supset U_q(\mathrm{so}(3))$ should exist.

An alternative approach to q-deformation of the orthogonal and pseudoorthogonal Lie algebras was proposed in [7],[8] and further developed in [9]-[11]. Such *nonstandard* q-analogue $U_q(\mathrm{so}(n,\mathbf{C}))$ of the Lie algebra $\mathrm{so}(n,\mathbf{C})$ is constructed not in terms of Chevalley basis (i.e., Cartan subalgebra and simple root elements, as it was done within standard approach of [1],[2]) but, starting from $\mathrm{so}(n,\mathbf{C})$ formulated as the complex associative algebra with $n-1$ generating elements $I_{21}, I_{32}, \ldots, I_{n,n-1}$. The *nonstandard* q-analogues $U_q(\mathrm{so}(n,\mathbf{C}))$ are formulated in a uniform fashion for all values $n \geq 3$, and guarantee validity of the canonical chain of embeddings

$$U_q(\mathrm{so}(n,\mathbf{C})) \supset U_q(\mathrm{so}(n-1,\mathbf{C})) \supset \cdots \supset U_q(\mathrm{so}(3,\mathbf{C})). \tag{1}$$

Due to this, one can attempt to develop a q-analogue of GT formalism. Other viable feature is the admittance by these $U_q(\mathrm{so}(n,\mathbf{C}))$ of all the noncompact real forms corresponding to those in classical case. In particular, validity of inclusions $U_q(\mathrm{so}(n,1)) \supset U_q(\mathrm{so}(n)) \supset U_q(\mathrm{so}(n-1)) \supset \ldots$ can be exploited in order to analyze infinite dimensional representations of q-deformed Lorentz algebras (see [10] where detailed study of class 1 representations of $U_q(\mathrm{so}(n,1))$ is presented).

The purpose of this paper is to provide, *along with detailed proof of validity*, exact matrix realization of finite dimensional irreducible representations of the nonstandard q-deformation $U_q(\mathrm{so}(n,\mathbf{C}))$ that directly correspond to the finite dimensional irreducible representations of $\mathrm{so}(n,\mathbf{C})$ given by highest weights with all integral or all half-integral components. Those representations, with an additional restriction on the deformation parameter q, yield irreducible infinitesimally unitary (or $*$-) representations of the 'compact' real form $U_q(\mathrm{so}_n)$.

2. Nonstandard q-deformed algebras $U_q(\mathrm{so}_n)$

According to [7]-[10], the nonstandard q-deformation $U_q(\mathrm{so}(n,\mathbf{C}))$ of the Lie algebra $\mathrm{so}(n,\mathbf{C})$ is given as a complex associative algebra with $n-1$ generating elements $I_{21}, I_{32}, \ldots,$ $I_{n,n-1}$ obeying the defining relations (denote $q+q^{-1} \equiv [2]_q$)

$$(2)\qquad I^2_{k,k-1}I_{k-1,k-2}+I_{k-1,k-2}I^2_{k,k-1}-[2]_q\ I_{k,k-1}I_{k-1,k-2}I_{k,k-1}=-I_{k-1,k-2},$$

$$(3)\qquad I^2_{k-1,k-2}I_{k,k-1}+I_{k,k-1}I^2_{k-1,k-2}-[2]_q\ I_{k-1,k-2}I_{k,k-1}I_{k-1,k-2}=-I_{k,k-1},$$

$$(4)\qquad [I_{i,i-1},I_{k,k-1}]=0 \qquad \text{if} \quad \mid i-k\mid>1,$$

where $i,k=2,3,...,n$. These relations are certainly true at $q=1$, i.e., within $\mathrm{so}(n,\mathbf{C})$. The real forms - compact $U_q(\mathrm{so}_n)$ and noncompact $U_q(\mathrm{so}_{n-1,1})$ - are singled out from the algebra $U_q(\mathrm{so}(n,\mathbf{C}))$ by imposing the $*$-structures

$$(5)\qquad I^*_{k,k-1}=-I_{k,k-1},\quad k=2,...,n,$$

and

$$(6)\qquad I^*_{k,k-1}=-I_{k,k-1},\quad k=2,...,n-1,\qquad I^*_{n,n-1}=I_{n,n-1},$$

respectively[2]. It is obvious that complex as well as compact and noncompact real q-deformed algebras so defined are compatible with the canonical reductions mentioned above. This enables one to apply a q-analogue of GT formalism in constructing representations. Finite dimensional representations T of the algebras $U_q(\mathrm{so}_n)$ are characterized by signatures completely analogous to highest weights of the algebras $\mathrm{so}(n)$ and described by means of action formulas for the operators $T(I_{k,k-1})$, $k=2,...,n$, which satisfy the relations (1)-(3) and the relations

$$(7)\qquad T(I_{k,k-1})^*=-T(I_{k,k-1}).$$

Finite dimensional representations of $U_q(\mathrm{so}_3)$, $U_q(\mathrm{so}_4)$ and infinite dimensional representations of their noncompact analogues $U_q(\mathrm{so}_{2,1})$, $U_q(\mathrm{so}_{3,1})$ were studied in [9],[10].

Remark. As pointed out in [9]-[10], the algebra $U_q(\mathrm{so}_n)$ for $n=3$, when presented in terms of *bilinear* defining relations (i.e., with q-deformed commutators) for I_{21} and I_{32} is isomorphic to the (cyclically symmetric, Cartesian) q-deformed algebra which was studied by Odesskii [13] and Fairlie [14] where some representations were also given. Classification of irreducible $*$-representations of the algebra $U_q(\mathrm{so}_3)$ is worked out in [15].

3. Representations of the algebras $U_q(\mathrm{so}_n)$

In this section we describe explicitly finite dimensional representations of the algebras $U_q(\mathrm{so}_n)$, $n\geq 3$ in the framework of a q-analogue of GT formalism. These are given by signatures - sets $\mathbf{m}_n$ consisting of $[\frac{n}{2}]$ components $m_{1,n},m_{2,n},...,m_{[\frac{n}{2}],n}$ (here $[\frac{n}{2}]$ denotes integer part of $\frac{n}{2}$) that satisfy the dominance condition, respectively, for $n=2p+1$ and $n=2p$:

$$(8a)\qquad m_{1,2p+1}\geq m_{2,2p+1}\geq ...\geq m_{p,2p+1}\geq 0,$$

$$(8b)\qquad m_{1,2p}\geq m_{2,2p}\geq ...\geq m_{p-1,2p}\geq |m_{p,2p}|.$$

[2] Other noncompact real forms $U_q(\mathrm{so}_{n-p,p})$, defined by their corresponding $*$-structures, are also possible [7],[10].

For a basis in representation space we exploit the q-analogue of GT basis [3],[4]. Its elements are labelled by GT schemes

$$\{\xi_n\} \equiv \left\{ \begin{array}{l} \mathbf{m}_n \\ \mathbf{m}_{n-1} \\ \dots \\ \mathbf{m}_2 \end{array} \right\} \equiv \{\mathbf{m}_n, \xi_{n-1}\} \equiv \{\mathbf{m}_n, \mathbf{m}_{n-1}, \xi_{n-2}\}, \tag{9}$$

where the components of $\mathbf{m}_n$ and $\mathbf{m}_{n-1}$ satisfy the "betweenness" conditions

$$m_{1,2p+1} \geq m_{1,2p} \geq m_{2,2p+1} \geq m_{2,2p} \geq ... \geq m_{p,2p+1} \geq m_{p,2p} \geq -m_{p,2p+1}, \tag{10a}$$

$$m_{1,2p} \geq m_{1,2p-1} \geq m_{2,2p} \geq m_{2,2p-1} \geq ... \geq m_{p-1,2p-1} \geq |m_{p,2p}|. \tag{10b}$$

Basis element defined by scheme $\{\xi_n\}$ is denoted as $|\{\xi_n\}\rangle$ or simply as $|\xi_n\rangle$.

We use standard denotion for q-number $[x] \equiv (q^x - q^{-x})/(q - q^{-1})$ corresponding to an integer or a half-integer x. Also, it is convenient to introduce the so-called l-coordinates

$$l_{j,2p+1} = m_{j,2p+1} + p - j + 1, \qquad l_{j,2p} = m_{j,2p} + p - j. \tag{11}$$

Infinitesimal operator $I_{2p+1,2p}$ of the representation, given by $\mathbf{m}_{2p+1}$, of $U_q(\mathrm{so}_{2p+1})$ acts upon GT basis elements, labelled by (9), according to (here $\beta \equiv \xi_{2p-1}$)

$$\begin{aligned} I_{2p+1,2p}|\mathbf{m}_{2p+1}, \mathbf{m}_{2p}, \beta\rangle = & \sum_{j=1}^{p} A_{2p}^{j}(\mathbf{m}_{2p})|\mathbf{m}_{2p+1}, \mathbf{m}_{2p}^{+j}, \beta\rangle \\ & - \sum_{j=1}^{p} A_{2p}^{j}(\mathbf{m}_{2p}^{-j})|\mathbf{m}_{2p+1}, \mathbf{m}_{2p}^{-j}, \beta\rangle \end{aligned} \tag{12}$$

and the operator $I_{2p,2p-1}$ of the representation, given by $\mathbf{m}_{2p}$, of $U_q(\mathrm{so}_{2p})$ acts as (here $\beta \equiv \xi_{2p-2}$)

$$\begin{aligned} I_{2p,2p-1}|\mathbf{m}_{2p}, \mathbf{m}_{2p-1}, \beta\rangle = & \sum_{j=1}^{p-1} B_{2p-1}^{j}(\mathbf{m}_{2p-1})|\mathbf{m}_{2p}, \mathbf{m}_{2p-1}^{+j}, \beta\rangle \\ & - \sum_{j=1}^{p-1} B_{2p-1}^{j}(\mathbf{m}_{2p-1}^{-j})|\mathbf{m}_{2p}, \mathbf{m}_{2p-1}^{-j}, \beta\rangle \\ & + i\ C_{2p-1}(\mathbf{m}_{2p-1})|\mathbf{m}_{2p}, \mathbf{m}_{2p-1}, \beta\rangle. \end{aligned} \tag{13}$$

In these formulas, $\mathbf{m}_n^{\pm j}$ means that the j-th component $m_{j,n}$ in signature $\mathbf{m}_n$ is to be replaced by $m_{j,n} \pm 1$; matrix elements A_{2p}^j, B_{2p-1}^j, C_{2p-1} are given by the expressions

$$\begin{aligned} A_{2p}^{j}(\xi_{2p+1}) = d(l_{j,2p}) \Bigg| & \frac{\prod_{i=1}^{p}[l_{i,2p+1} + l_{j,2p}][l_{i,2p+1} - l_{j,2p} - 1]}{\prod_{i\neq j}^{p}[l_{i,2p} + l_{j,2p}][l_{i,2p} - l_{j,2p}]} \\ & \times \frac{\prod_{i=1}^{p-1}[l_{i,2p-1} + l_{j,2p}][l_{i,2p-1} - l_{j,2p} - 1]}{\prod_{i\neq j}^{p}[l_{i,2p} + l_{j,2p} + 1][l_{i,2p} - l_{j,2p} - 1]} \Bigg|^{\frac{1}{2}} \end{aligned} \tag{14}$$

with

$$d(l_{j,2p}) \equiv \left(\frac{[l_{j,2p}][l_{j,2p}+1]}{[2l_{j,2p}][2l_{j,2p}+2]} \right)^{\frac{1}{2}} \tag{15}$$

and

$$B^j_{2p-1}(\xi_{2p}) = \left| \frac{\prod_{i=1}^{p}[l_{i,2p}+l_{j,2p-1}][l_{i,2p}-l_{j,2p-1}]}{[2l_{j,2p-1}+1][2l_{j,2p-1}-1]\prod_{i\neq j}^{p-1}[l_{i,2p-1}+l_{j,2p-1}][l_{i,2p-1}-l_{j,2p-1}]} \right.$$

$$\left. \times \frac{\prod_{i=1}^{p-1}[l_{i,2p-2}+l_{j,2p-1}][l_{i,2p-2}-l_{j,2p-1}]}{[l_{j,2p-1}]^2 \prod_{i\neq j}^{p-1}[l_{i,2p-1}+l_{j,2p-1}-1][l_{i,2p-1}-l_{j,2p-1}-1]} \right|^{\frac{1}{2}}, \tag{16}$$

$$C_{2p-1}(\xi_{2p}) = \frac{\prod_{s=1}^{p}[l_{s,2p}]\prod_{s=1}^{p-1}[l_{s,2p-2}]}{\prod_{s=1}^{p-1}[l_{s,2p-1}][l_{s,2p-1}-1]}. \tag{17}$$

From (17) it is seen that C_{2p-1} in (13) identically vanishes if $m_{p,2p} = l_{p,2p} = 0$.

Remark. Matrix elements $B^j_{2p-1}(\xi)$ and $C_{2p-1}(\xi)$ are nothing but "minimal" deformation of their classical (i.e., $q = 1$) counterparts. On the other hand, because of appearance of the nontrivial factor $d(l_{j,2p})$ (which replaces the constant multiplier $\frac{1}{2}$ of classical case) in $A^j_{2p}(\xi)$, these matrix elements deviate from "minimal" deformation.

Proposition. *Let $q^N \neq 1$, $N \in \mathbf{Z}\backslash\{0\}$, $q \in \mathbf{C}$. Representation operators $T_{\mathbf{m}_n}(I_{k,k-1})$, $k = 2, ..., n$, of the representation $T_{\mathbf{m}_n}$, characterized by the signature $\mathbf{m}_n$ with all integral or all half-integral components satisfying (8), given in the q-analogue of GT basis (9),(10) by the action formulas (12)-(17) satisfy the defining relations (2)-(4) of the algebra $U_q(\mathrm{so}(n,\mathbf{C}))$ for both even and odd n and define finite dimensional irreducible representations of this algebra. If in addition $q = e^h$ or $q = e^{ih}$, $h \in \mathbf{R}$, these operators satisfy the condition (7) and thus provide (infinitesimally unitary or) $*$-representations of $U_q(\mathrm{so}_n)$.*

Proof. First, we prove that the representation operators $T_{\mathbf{m}_n}(I_{k,k-1})$, $k = 2, \dots, n$ of the algebra $U_q(\mathrm{so}(n,\mathbf{C}))$ given in the q-analog of GT basis by explicit formulas (12)-(17) indeed satisfy the defining relations (2)-(4). The fact that any pair of operators $T_{\mathbf{m}_n}(I_{k,k-1})$ and $T_{\mathbf{m}_n}(I_{i,i-1})$,[3] where $|i-k| > 1$ do commute (cf.(4)) follows from the structure of their action formulas (e.g. in the "nearest" case of $i-k = 2$ the operator $I_{k,k-1}$ changes components of signatures $\mathbf{l}_{k-1}$ while its matrix elements do not depend on $\mathbf{l}_{k+1}$ affected by the action of $I_{k+2,k+1}$, and vice versa). Now let us prove the relations (2)-(3) for $I_{k,k-1}$ taken for sequential (odd and even) values of k, say $2p+1$ and $2p$:

$$\mathbf{I}. \qquad I^2_{2p+1,2p}I_{2p,2p-1} + I_{2p,2p-1}I^2_{2p+1,2p} - [2]\ I_{2p+1,2p}I_{2p,2p-1}I_{2p+1,2p} = -I_{2p,2p-1};$$

$$\mathbf{II}. \qquad I^2_{2p,2p-1}I_{2p+1,2p} + I_{2p+1,2p}I^2_{2p,2p-1} - [2]\ I_{2p,2p-1}I_{2p+1,2p}I_{2p,2p-1} = -I_{2p+1,2p};$$

[3] For convenience, below by the symbol $I_{k,k-1}$ we mean also the corresponding representation operator $T_{\mathbf{m}_n}(I_{k,k-1})$.

$$\textbf{III}.\quad I^2_{2p,2p-1}I_{2p-1,2p-2}+I_{2p-1,2p-2}I^2_{2p,2p-1}-[2]\ I_{2p,2p-1}I_{2p-1,2p-2}I_{2p,2p-1} = -I_{2p-1,2p-2};$$

$$\textbf{IV}.\quad I^2_{2p-1,2p-2}I_{2p,2p-1}+I_{2p,2p-1}I^2_{2p-1,2p-2}-[2]\ I_{2p-1,2p-2}I_{2p,2p-1}I_{2p-1,2p-2} = -I_{2p,2p-1}.$$

Action of left and right hand sides of each of these equalities upon some generic basis vector $\mid\xi\rangle$ of the representation space $\mathcal{V}_{\mathbf{m}_n}$ produces, besides the initial basis vector (if any), also a number of other basis vectors. It is necessary to examine all the encountered resulting basis vectors for each of the relations (I)-(IV).

Case of (I)

The list of resulting basis vectors for (I) is as follows.
Action of RHS of (I) on the vector $\mid\xi\rangle$ leads to vectors :

(I.1.a) $\mid\xi\rangle$ (unchanged vector): 1 vector

(I.1.b) $|l_{j,2p-1}\pm 1\rangle$: $2(p-1)$ vectors

Action of LHS of (I) on the same vector $\mid\xi\rangle$ leads to vectors :

(I.1.a) $\mid\xi\rangle$ (unchanged vector): 1 vector

(I.1.b) $|l_{j,2p-1}\pm 1\rangle$: $2(p-1)$ vectors

(I.2.a) $|l_{j',2p}\pm 2\rangle$: $2p$ vectors

(I.2.b) $|l_{j',2p}\pm 2; l_{j,2p-1}\pm 1\rangle$: $4p(p-1)$ vectors

(I.3.a) $|l_{j',2p}\pm 1; l_{j'',2p}\pm 1\rangle$: $2p(p-1)$ vectors

(I.3.b) $|l_{j',2p}\pm 1; l_{j'',2p}\pm 1; l_{j,2p-1}\pm 1\rangle$: $4p(p-1)(p-1)$ vectors

(only the changed component(s) off those labelling the vectors are indicated; the signs $\pm$ in the last three items take their values independently).

Comparison of expressions at the same fixed vector $\mid\tilde{\xi}\rangle$ in the LHS and RHS yields some particular relation (in what follows, we use for it the term *relation corresponding to the vector* $\mid\tilde{\xi}\rangle$). The proof of equality (I) will be achieved if all the possible relations (obtained from examination of the complete list of the encountered basis vectors for this equality) are proven to hold identically. For every type of the vectors in the list given above, we prove validity of relation corresponding to one typical representative; the proof of relations corresponding to the other vectors of this same type is completely analogous. For instance, concerning the type (I.3.a) we prove the relation corresponding to typical representative $|l_{j',2p}+1; l_{j'',2p}+1\rangle$.

The vectors (I.2.a), (I.2.b), (I.3.a), (I.3.b) appear only in the LHS. Hence, the relations which correspond to them have zero right hand sides; verification of these relations proceeds in straightforward way. Let us start with these 4 cases.

Relations corresponding to (I.2.a)

To typical vector $|l_{j',2p}+2\rangle$ there corresponds the relation

$$C_{2p-1}(l_{j',2p}+2)+C_{2p-1}(l_{j',2p})-[2]C_{2p-1}(l_{j',2p}+1)=0.$$

On the base of the expression (17) for C_{2p-1} it is reduced to the equality $[l_{j',2p}+2]+[l_{j',2p}]-[2][l_{j',2p}+1]=0$, which coincides with the well-known identity valid for any three successive q-numbers:

$$[x+1]=[2][x]-[x-1]. \tag{18}$$

Relations corresponding to (I.2.b)

The relation corresponding to typical vector $|l_{j',2p}+2; l_{j,2p-1}+1\rangle$ is

$$A^{j'}_{2p}(l_{j',2p}+1)A^{j'}_{2p}B^{j}_{2p-1}\Big(\frac{B^{j}_{2p-1}(l_{j',2p}+2)}{B^{j}_{2p-1}}+\frac{A^{j'}_{2p}(l_{j',2p}+1;l_{j,2p-1}+1)}{A^{j'}_{2p}(l_{j',2p}+1)}\frac{A^{j'}_{2p}(l_{j,2p-1}+1)}{A^{j'}_{2p}}$$

$$-[2]\frac{B^{j}_{2p-1}(l_{j',2p}+1)}{B^{j}_{2p-1}}\frac{A^{j'}_{2p}(l_{j',2p}+1;l_{j,2p-1}+1)}{A^{j'}_{2p}(l_{j',2p}+1)}\Big)=0.$$

Using explicit forms of $A^{j'}_{2p}$ and B^{j}_{2p-1}, after simple algebra it reduces to the identity

$$[l_{j',2p}-l_{j,2p-1}+2]+[l_{j',2p}-l_{j,2p-1}]-[2][l_{j',2p}-l_{j,2p-1}+1]=0 \tag{19}$$

which is of the same type as (18).

Relations corresponding to (I.3.a)

The relation corresponding to typical vector $|l_{j',2p}+1; l_{j'',2p}+1\rangle$ has the following form :

$$A^{j'}_{2p}(l^{+j''}_{2p})A^{j''}_{2p}\Big(1+\frac{C_{2p-1}(l^{+j'}_{2p};l^{+j''}_{2p})}{C_{2p-1}}-[2]\frac{C_{2p-1}(l^{+j''}_{2p})}{C_{2p-1}}\Big)$$

$$+A^{j''}_{2p}(l^{+j'}_{2p})A^{j'}_{2p}\Big(1+\frac{C_{2p-1}(l^{+j'}_{2p};l^{+j''}_{2p})}{C_{2p-1}}-[2]\frac{C_{2p-1}(l^{+j'}_{2p})}{C_{2p-1}}\Big)=0. \tag{20}$$

Using the explicit formula (17) for C_{2p-1} we transform the expressions multiplying $A^{j'}_{2p}(l^{+j''}_{2p})A^{j''}_{2p}$ and $A^{j''}_{2p}(l^{+j'}_{2p})A^{j'}_{2p}$ to $-[l_{j',2p}-l_{j'',2p}-1]$ and $[l_{j',2p}-l_{j'',2p}+1]$ respectively. Thus, the relation (20) can be rewritten as

$$A^{j'}_{2p}A^{j''}_{2p}\Big(-[l_{j',2p}-l_{j'',2p}-1]\frac{A^{j'}_{2p}(l^{+j''}_{2p})}{A^{j'}_{2p}}+[l_{j',2p}-l_{j'',2p}+1]\frac{A^{j''}_{2p}(l^{+j'}_{2p})}{A^{j''}_{2p}}\Big)=0,$$

and the latter can be verified easily with the use of explicit expressions for A^{j}_{2p} .

Relations corresponding to (I.3.b)

The proof in this case can be carried out in a manner similar to that of the case (I.3.a).

Relation corresponding to (I.1.a)

Next, consider the relation which corresponds to the unchanged vector (I.1.a), i.e.,

$$\sum_{r=1}^{p}\Bigg\{(A^{r}_{2p})^2\Bigg(2-[2]\frac{C_{2p-1}(l^{+r}_{2p})}{C_{2p-1}}\Bigg)$$

$$+(A^{r}_{2p-1}(l^{-r}_{2p-1}))^2\Bigg(2-[2]\frac{C_{2p-1}(l^{-r}_{2p})}{C_{2p-1}}\Bigg)\Bigg\}=1. \tag{21}$$

Let us introduce ϕ^r defined as

$$\phi^r(\{l_{1,2p+1};l_{1,2p};l_{1,2p-1}\},\ldots,\{l_{r,2p+1};l_{r,2p};l_{r,2p-1}\},\ldots,\{l_{p,2p+1};l_{p,2p};.\}) \tag{22}$$

$$\equiv \frac{\prod_{s=1}^{p} f(l_{s,2p+1}; l_{r,2p}) \prod_{s=1}^{p-1} f(l_{s,2p-1}; l_{r,2p})}{\prod_{s\neq r}^{p} f(l_{s,2p}; l_{r,2p}) f(l_{s,2p}+1; l_{r,2p})}$$

where $f(x;y) \equiv [x+y][x-y-1]$.

Then, using the expression (17) for C_{2p-1} and relations

$$2-[2]\frac{[l_{r,2p}+1]}{[l_{r,2p}]} = -\frac{[2l_{r,2p}+2]}{[l_{r,2p}+1][l_{r,2p}]},$$

$$2-[2]\frac{[l_{r,2p}-1]}{[l_{r,2p}]} = \frac{[2l_{r,2p}-2]}{[l_{r,2p}-1][l_{r,2p}]}$$

(the latter two can be easily verified by means of the identity $[x+1]-[x-1]=[2x]/[x]$), we can rewrite (21) in the form which coincides with the relation (A.1) of Appendix if we set

$$\phi^r(..., \{l_{r,2p+1}; l_{r,2p}; l_{r,2p-1}\}, ...) \equiv \frac{[2l_{r,2p}][2l_{r,2p}+2]}{[l_{r,2p}][l_{r,2p}+1]}(A^r_{2p}(l_{r,2p}))^2. \tag{23}$$

Relations corresponding to (I.1.b)

Let us verify the following relation corresponding to typical vector $|l_{j,2p-1}+1\rangle$:

$$\sum_{r=1}^{p}\Bigg\{(A^r_{2p})^2\left(1+\frac{(A^r_{2p}(l^{+j}_{2p-1}))^2}{(A^r_{2p})^2}-[2]\frac{B^j_{2p-1}(l^{+r}_{2p})}{B^j_{2p-1}}\frac{A^r_{2p}(l^{+j}_{2p-1})}{A^r_{2p}}\right) \tag{24}$$

$$+(A^r_{2p}(l^{-r}_{2p}))^2\left(1+\frac{(A^r_{2p}(l^{-r}_{2p};l^{+j}_{2p-1}))^2}{(A^r_{2p}(l^{-r}_{2p}))^2}-[2]\frac{B^j_{2p-1}(l^{-r}_{2p})}{B^j_{2p-1}}\frac{A^r_{2p}(l^{-r}_{2p};l^{+j}_{2p-1})}{A^r_{2p}(l^{-r}_{2p})}\right)\Bigg\} = 1.$$

Evaluating the expression at $(A^r_{2p})^2$ we have $[2l_{r,2p}+2]/([l_{j,2p-1}-l_{r,2p}-1][l_{j,2p-1}+l_{r,2p}])$ (the q-number identities $[x][y]=[(x+y)/2]^2-[(x-y)/2]^2$ and $[x]^2-[y]^2=[x+y][x-y]$ were utilized). As a result, the term containing $(A^r_{2p})^2$ in the first line of (24) takes the form

$$\frac{1}{[2l_{r,2p}]}[l_{r,2p}][l_{r,2p}+1]\frac{\phi^r}{[l_{j,2p-1}-l_{r,2p}-1][l_{j,2p-1}+l_{r,2p}]} \tag{25}$$

(here the correspondence (23) for ϕ^r and $(A^r_{2p})^2$) was used).

Notice that in the last factor (i.e., in the ratio) dependence on $l_{j,2p-1}$ in fact cancels out. However, $[l_{r,2p}][l_{r,2p}+1] \equiv -([l_{j,2p-1}-l_{r,2p}-1][l_{j,2p-1}+l_{r,2p}])|_{l_{j,2p-1}=0}$ and thus the expression (25) can be rewritten as

$$-\frac{1}{[2l_{r,2p}]}\phi^r\Big|_{l_{j,2p-1}=0}.$$

Analogously, it can be shown that the expression at $(A^r_{2p}(l^{-r}_{2p}))^2$ in the relation (24) is equal to $-[2l_{r,2p}-2]/([l_{j,2p-1}-l_{r,2p}][l_{j,2p-1}+l_{r,2p}-1])$ and hence the whole term containing $(A^r_{2p}(l^{-r}_{2p}))^2$ in the second line of (24) can be rewritten as

$$\frac{1}{[2l_{r,2p}]}\phi^r(l^{-r}_{2p})\Big|_{l_{j,2p-1}=0}.$$

Thus, the initial relation (24) is reduced to a special case[4] (with $l_{j,2p-1}=0$) of the relation (A.1) proved for arbitrary $l_{j,2p-1}$ in the Appendix.

Case of (II)

The proof given above (for the case of equality (I)) by analogy carries over to the remaining cases of equalities (II),(III),(IV). Let us prove the equality (II).
The action of RHS in (II) on the vector $\mid\xi\rangle$ leads to the vectors :

(II.1) $|l_{j,2p}\pm 1\rangle$: $2p$ vectors

The action of LHS in (II) on the same vector leads to the vectors :

(II.1) $|l_{j,2p}\pm 1\rangle$: $2p$ vectors

(II.2) $|l_{j',2p-1}\pm 2; l_{j,2p}\pm 1\rangle$: $4p(p-1)$ vectors

(II.3) $|l_{j',2p-1}\pm 1; l_{j'',2p-1}\pm 1; l_{j,2p}\pm 1\rangle$: $4p(p-1)(p-2)$ vectors

(II.4) $|l_{j',2p-1}\pm 1; l_{j,2p}\pm 1\rangle$: $4p(p-1)$ vectors

(only the changed component(s) off those labelling the vectors are indicated; the signs $\pm$ in the last three items take their values independently).

Let us verify the relations, which can be obtained by equating the coefficients at the same vectors from the LHS and RHS .

Relations corresponding to (II.2), (II.3)

Relations corresponding to (II.2) and (II.3) can be verified in a way similar to the cases (I.2.b) and (I.3.b) respectively.

Relations corresponding to (II.4)

Let us consider relation which corresponds to typical vector $|l_{j',2p-1}+1; l_{j,2p}+1\rangle$. Its explicit form is

$$C_{2p-1}A^{j}_{2p}B^{j'}_{2p-1}\left(\frac{A^{j}_{2p}(l^{+j'}_{2p-1})}{A^{j}_{2p}}+\frac{B^{j'}_{2p-1}(l^{+j}_{2p})}{B^{j'}_{2p-1}}\frac{C_{2p-1}(l^{+j}_{2p})}{C_{2p-1}}-[2]\frac{B^{j'}_{2p-1}(l^{+j}_{2p})}{B^{j'}_{2p-1}}\right)$$

$$+C_{2p-1}(l^{+j'}_{2p-1})A^{j}_{2p}B^{j'}_{2p-1}\left(\frac{A^{j}_{2p}(l^{+j'}_{2p-1})}{A^{j}_{2p}}+\frac{B^{j'}_{2p-1}(l^{+j}_{2p})}{B^{j'}_{2p-1}}\frac{C_{2p-1}(l^{+j}_{2p},l^{+j'}_{2p-1})}{C_{2p-1}(l^{+j'}_{2p-1})}\right.$$

$$\left.-[2]\frac{A^{j}_{2p}(l^{+j'}_{2p-1})}{A^{j}_{2p}}\frac{C_{2p-1}(l^{+j}_{2p},l^{+j'}_{2p-1})}{C_{2p-1}(l^{+j'}_{2p-1})}\right)=0. \tag{26}$$

After evaluation (which uses explicit form of $A^{j}_{2p}, B^{j'}_{2p-1}, C_{2p-1}$ and some q-number identities) we obtain that the expression which gives the coefficient at $C_{2p-1}A^{j}_{2p}B^{j'}_{2p-1}$ is

$$-\frac{[l_{j',2p-1}-1]}{[l_{j,2p}]([l_{j',2p-1}-l_{j,2p}-1][l_{j',2p-1}-l_{j,2p}])^{1/2}}$$

while the coefficient at $C_{2p-1}(l^{+j'}_{2p-1})A^{j}_{2p}B^{j'}_{2p-1}$ is

$$\frac{[l_{j',2p-1}+1]}{[l_{j,2p}]([l_{j',2p-1}-l_{j,2p}-1][l_{j',2p-1}-l_{j,2p}])^{1/2}}.$$

[4]Although the value $l_{j,2p-1}=0$ contradicts (8) and (10), the identity (A.1) nevertheless remains true for that value in this special case of (I.1.b).

Using these results, it is easy to verify validity of (26).

Relations corresponding to (II.1)

Let us consider the relation which corresponds to a typical vector $|l_{j,2p}+1\rangle$. Its explicit form is as follows :

$$\sum_{r=1}^{p-1}\Bigg\{(B^r_{2p-1})^2\left(1+\frac{(B^r_{2p-1}(l^{+j}_{2p}))^2}{(B^r_{2p-1})^2}-[2]\frac{B^r_{2p-1}(l^{+j}_{2p})}{B^r_{2p-1}}\frac{A^j_{2p}(l^{+r}_{2p-1})}{A^j_{2p}}\right)$$

$$+(B^r_{2p-1}(l^{-r}_{2p-1}))^2\left(1+\frac{(B^r_{2p-1}(l^{-r}_{2p-1};l^{+j}_{2p}))^2}{(B^r_{2p-1}(l^{-r}_{2p-1}))^2}-[2]\frac{B^r_{2p-1}(l^{-r}_{2p-1};l^{+j}_{2p})}{B^r_{2p-1}(l^{-r}_{2p-1})}\frac{A^j_{2p}(l^{-r}_{2p-1})}{A^j_{2p}}\right)\Bigg\}$$

$$+(C_{2p-1})^2\left(1+\frac{(C_{2p-1}(l^{+j}_{2p}))^2}{(C_{2p-1})^2}-[2]\frac{C_{2p-1}(l^{+j}_{2p})}{C_{2p-1}}\right)=1. \tag{27}$$

The coefficients in (27) at $(B^r_{2p-1})^2$, $(B^r_{2p-1}(l^{-r}_{2p-1}))^2$ and $(C_{2p-1})^2$ can be reduced, respectively, to the expressions

$$\frac{[2l_{r,2p-1}+1]}{[l_{j,2p}-l_{r,2p-1}][l_{j,2p}+l_{r,2p-1}]},\quad -\frac{[2l_{r,2p-1}-3]}{[l_{j,2p}-l_{r,2p-1}+1][l_{j,2p}+l_{r,2p-1}-1]},\quad \frac{1}{[l_{j,2p}]^2}.$$

Then the relation (27) transforms into

$$\sum_{r=1}^{p-1}\frac{1}{[2l_{r,2p-1}-1]}\Bigg\{\frac{[2l_{r,2p-1}+1][2l_{r,2p-1}-1]}{[l_{j,2p}-l_{r,2p-1}][l_{j,2p}+l_{r,2p-1}]}(B^r_{2p-1})^2$$

$$-\frac{[2l_{r,2p-1}-1][2l_{r,2p-1}-3]}{[l_{j,2p}-l_{r,2p-1}+1][l_{j,2p}+l_{r,2p-1}-1]}(B^r_{2p-1}(l^{-r}_{2p-1}))^2\Bigg\}+\frac{1}{[l_{j,2p}]^2}(C_{2p-1})^2=1. \tag{28}$$

Observe that the dependence on $l_{j,2p}$ in LHS of (28) cancels out.

Let us show that (28) is a particular case of the relation (A.1) proved in the Appendix. To this end, we make replacements[5]

$l_{s,2p-1}\to l_{s,2p-2}+\frac{1}{2}$, $\quad s=1,\dots,p-1$;
$l_{s,2p}\to l_{s,2p-1}-\frac{1}{2}$, $\quad s=1,\dots,p-1$;
$l_{s,2p+1}\to l_{s,2p}+\frac{1}{2}$, $\quad s=1,\dots,p, \quad s\neq j$;
$l_{j,2p+1}\to \frac{3}{2}$; $\quad l_{p,2p}\to -\frac{1}{2}$;

in the LHS of (A.1). These replacements imply (recall that ϕ^r is defined in (22)):

$$\phi^r\to -\frac{[2l_{r,2p-1}+1][2l_{r,2p-1}-1]}{[l_{j,2p}-l_{r,2p-1}][l_{j,2p}+l_{r,2p-1}]}(B^r_{2p-1})^2,\qquad \frac{1}{[2l_{r,2p}]}\to\frac{1}{[2l_{r,2p-1}-1]}$$

for $r=1,\dots,p-1$, and

$$\phi^p\to\frac{1}{[l_{j,2p}]^2}(C_{2p-1})^2,\qquad \phi^p(l^{-p}_{2p})\to 0,\qquad \frac{1}{[2l_{p,2p}]}\to -1.$$

Therefore, $\Phi(\dots)=1$ from (A.1) goes into (28). Thus, we have shown that the relation (28) holds simultaneously with (A.1) (which is proved in the Appendix).

[5] Although the value $\frac{3}{2}$ for $l_{j,2p+1}$ contradicts (8) and (10), the identity (A.1) nevertheless remains true for that value (formal for $j<p$) in this special case of (II.1).

Case of (IV)

The proof of relations which appear in conjunction with (IV) is completely analogous to that of (I). The only difference consists in formal replacements $l_{j,2p-1} \leftrightarrow l_{j,2p+1}$, and then $p \to p-1$. For instance, to prove the relation corresponding to the vector $|l_{j',2p-2}+2; l_{j,2p-1}+1\rangle$, one uses the identity $[l_{j',2p-2}-l_{j,2p-1}+2]+[l_{j',2p-2}-l_{j,2p-1}]-[2][l_{j',2p-2}-l_{j,2p-1}+1]=0$, which can be obtained from the identity (19) with the replacements just pointed out.

Case of (III)

The proof of relations corresponding to (vectors arising in conjunction with) (III) proceeds in complete analogy to that of (II) with the appropriate replacements $l_{j,2p} \leftrightarrow l_{j,2p-2}$ being done.

Finite dimensionality, irreducibility, $*$-property

Concerning proof of finite dimensionality and irreducibility of the $U_q(\mathrm{so}_n)$ representations presented in this paper, all things go through in complete analogy to the classical ($q=1$) case for the considered case of $\mathbf{m}_n$ with all integral or all half-integral components and the restriction $q^N \neq 1$. Finite dimensionality is verified on the base of explicit formulas for the representation matrix elements, with the account of the q-number property that, for real x and $q^N \neq 1$, the equality $[x]=0$ holds only if $x=0$. Thus, no new zeros for the matrix elements A^j_{2p}, B^j_{2p-1}, and C_{2p-1} can appear besides zeros appearing in $q=1$ case (and corresponding to limiting values in (10)). In the proof of irreducibility it is important to stress that for reals a, b, and $q^N \neq 1$, the equality $[a]=[b]$ cannot hold for $a \neq b$. With this property in mind, it is enough to trace the arguments of the proof in classical case.

The fact that the representation operators at $q=e^h$ or $q=e^{\mathrm{i}h}$, $h \in \mathbf{R}$, satisfy the $*$-relation (7) and thus define infinitesimally unitary (or $*$-) representations of the algebra $U_q(\mathrm{so}_n)$ is easily verified if one takes in account explicit matrix elements (14)-(17). $\square$

4. Summary and Outlook

The results presented above demonstrate unambiguously applicability of the (q-analogue of) Gel'fand–Tsetlin formalism for developing representation theory of the *nonstandard* q-deformed algebras $U_q(\mathrm{so}_n)$, contrary to the opinion existing in the literature, see [16],[17]. Within nonstandard q-deformation, one is able to construct the q-analogues of most general finite dimensional irreducible so_n representations (remark that so far, only symmetric tensor representations have been constructed [18] for the standard deformations $U_q(\mathrm{B}_r)$ and $U_q(\mathrm{D}_r)$). Moreover, it is possible to make use of such an effective formalism in order to construct and analyze infinite dimensional representations of the noncompact counterparts $U_q(\mathrm{so}_{n-1,1})$ of the orthogonal q-algebras. The same concerns q-deformed inhomogeneous (Euclidean) algebras $U_q(\mathrm{iso}_n)$ for which the chain $U_q(\mathrm{iso}_n) \supset U_q(\mathrm{so}_n) \supset U_q(\mathrm{so}_{n-1}) \supset \cdots \supset U_q(\mathrm{so}_3)$ is valid, as well as the chain $U_q(\mathrm{iso}_n) \supset U_q(\mathrm{iso}_{n-1}) \supset U_q(\mathrm{iso}_{n-2}) \supset \cdots \supset U_q(\mathrm{iso}_2)$. For the case of $U_q(\mathrm{iso}_n)$, the class 1 representations were studied in [19], and more general representations in [20].

Analysis of representations of nonstandard q-deformed algebras $U_q(\mathrm{so}_n)$ in the situation when the numbers characterizing representations are not necessarily integers or half-integers, as well as in the case of q equal to a root of unity, will be presented elsewhere.

Appendix

The aim of this appendix is to prove an important identity.

Proposition A. *The equality*

$$\Phi(\{l_{1,2p+1}; l_{1,2p}; l_{1,2p-1}\}, \ldots, \{l_{p-1,2p+1}; l_{p-1,2p}; l_{p-1,2p-1}\}, \{l_{p,2p+1}; l_{p,2p}; .\})$$

$$(A.1)\qquad \equiv \sum_{r=1}^{p} \frac{1}{[2l_{r,2p}]}\Big(-\phi^r(...,\{l_{r,2p+1};l_{r,2p};l_{r,2p-1}\},...)$$

$$+\phi^r(...,\{l_{r,2p+1};l_{r,2p}-1;l_{r,2p-1}\},...)\Big) = 1$$

with ϕ^r and $f(x;y)$ defined in (22) holds identically if the l-coordinates (see (11)) $l_{r,2p+1}; l_{r,2p}; l_{r,2p-1}$ are consistent with inequalities (8),(10).

Proof. Let us show that this statement for arbitrary $l_{1,2p+1}$ is a consequence of the statement reformulated for $l_{1,2p+1} = l_{1,2p}+1$ (if $l_{1,2p+1} = l_{1,2p}+1$ from the very beginning, the procedure that follows is not necessary). Using the identity $f(x;z)-f(y;z) = [x+z][x-z-1]-[y+z][y-z-1] = [x+y-1][x-y] = f(x;y-1)$ and its special case $f(l_{1,2p+1};l_{r,2p}) - f(l_{1,2p}+1;l_{r,2p}) = f(l_{1,2p+1};l_{1,2p})$ we get the following relations:

$$\Phi(\{l_{1,2p+1};l_{1,2p};l_{1,2p-1}\},\dots) - \Phi(\{l_{1,2p}+1;l_{1,2p};l_{1,2p-1}\},\dots)$$

$$(A.2)\qquad = f(l_{1,2p+1};l_{1,2p})\Phi(\{.;l_{1,2p};l_{1,2p-1}\},\dots),$$

$$\Phi(\{l_{1,2p}+1;l_{1,2p};l_{1,2p-1}\},\dots) - \Phi(\{l_{1,2p};l_{1,2p};l_{1,2p-1}\},\dots)$$

$$(A.3)\qquad = [2l_{1,2p}]\Phi(\{.;l_{1,2p};l_{1,2p-1}\},\dots).$$

Here "." in place of $l_{1,2p+1}$ in $\Phi(\dots)$ means that all the multipliers depending on $l_{1,2p+1}$ are omitted. Note that $\Phi(\{l_{1,2p};l_{1,2p};l_{1,2p-1}\},...)$ is well-defined as a formal expression in its variables, although the value of $l_{1,2p+1}$ equal to $l_{1,2p}$ in $\Phi(\{l_{1,2p};l_{1,2p};l_{1,2p-1}\},...)$ is excluded by inequalities (10).

It is possible to show by direct verification the validity of the identity

$$(A.4)\qquad \Phi(\{l_{1,2p};l_{1,2p};l_{1,2p-1}\},\dots) = \Phi(\{l_{1,2p}+1;l_{1,2p};l_{1,2p-1}\},\dots)|_{l_{1,2p}\to l_{1,2p}+1}.$$

We have in LHS of (A.4)

$$\phi^1(\{l_{1,2p};l_{1,2p};l_{1,2p-1}\},\dots) = \frac{\prod_{s=2}^{p} f(l_{s,2p+1};l_{1,2p})\prod_{s=1}^{p-1} f(l_{s,2p-1};l_{1,2p})f(l_{1,2p};l_{1,2p})}{\prod_{s=2}^{p} f(l_{s,2p};l_{1,2p})f(l_{s,2p}+1;l_{1,2p})},$$

$$\phi^1(\{l_{1,2p};l_{1,2p}-1;l_{1,2p-1}\},\dots) = 0,$$

$$\phi^r(\{l_{1,2p};l_{1,2p};l_{1,2p-1}\},\dots\{l_{r,2p};l_{r,2p};l_{r,2p-1}\},\dots)$$

$$= \frac{\prod_{s=2}^{p} f(l_{s,2p+1};l_{r,2p})\prod_{s=1}^{p-1} f(l_{s,2p-1};l_{r,2p})f(l_{1,2p};l_{r,2p})}{\prod_{s\geq 2,s\neq r}^{p} f(l_{s,2p};l_{r,2p})f(l_{s,2p}+1;l_{r,2p})f(l_{1,2p};l_{r,2p})f(l_{1,2p}+1;l_{r,2p})},$$

and in RHS of (A.4)

$$\phi^1(\{l_{1,2p}+1;l_{1,2p};l_{1,2p-1}\},\dots) = 0 \xrightarrow{l_{1,2p}\to l_{1,2p}+1} 0,$$

$$\phi^1(\{l_{1,2p}+1;l_{1,2p}-1;l_{1,2p-1}\},\dots)$$

$$= \frac{\prod_{s=2}^{p} f(l_{s,2p+1};l_{1,2p}-1)\prod_{s=1}^{p-1} f(l_{s,2p-1};l_{1,2p}-1)f(l_{1,2p}+1;l_{1,2p}-1)}{\prod_{s=2}^{p} f(l_{s,2p};l_{1,2p}-1)f(l_{s,2p}+1;l_{1,2p}-1)}$$

$$\xrightarrow{l_{1,2p}\to l_{1,2p}+1} \frac{\prod_{s=2}^{p} f(l_{s,2p+1};l_{1,2p})\prod_{s=1}^{p-1} f(l_{s,2p-1};l_{1,2p}) f(l_{1,2p}+2;l_{1,2p})}{\prod_{s=2}^{p} f(l_{s,2p};l_{1,2p}) f(l_{s,2p}+1;l_{1,2p})},$$

$$\begin{aligned}&\phi^r(\{l_{1,2p}+1;l_{1,2p};l_{1,2p-1}\},\ldots,\{l_{r,2p};l_{r,2p};l_{r,2p-1}\},\ldots)\\ &\quad = \frac{\prod_{s=2}^{p} f(l_{s,2p+1};l_{r,2p})\prod_{s=1}^{p-1} f(l_{s,2p-1};l_{r,2p}) f(l_{1,2p}+1;l_{r,2p})}{\prod_{s\geq 2,s\neq r}^{p} f(l_{s,2p};l_{r,2p}) f(l_{s,2p}+1;l_{r,2p}) f(l_{1,2p};l_{r,2p}) f(l_{1,2p}+1;l_{r,2p})}\\ &\quad \xrightarrow{l_{1,2p}\to l_{1,2p}+1} \frac{\prod_{s=2}^{p} f(l_{s,2p+1};l_{r,2p})\prod_{s=1}^{p-1} f(l_{s,2p-1};l_{r,2p})}{\prod_{s\geq 2,s\neq r}^{p} f(l_{s,2p};l_{r,2p}) f(l_{s,2p}+1;l_{r,2p}) f(l_{1,2p}+1;l_{r,2p})}.\end{aligned}$$

Using $f(l_{1,2p};l_{1,2p}) = -[2l_{1,2p}]$ and $f(l_{1,2p}+2;l_{1,2p}) = [2l_{1,2p}+2]$ we can verify (A.4) by comparing the terms in LHS and RHS of (A.4) .

Our assumption that statement (A.1) is correct for $l_{1,2p+1} = l_{1,2p}+1$ together with (A.4) imply that LHS of (A.3) is zero and therefore in the case of nonvanishing $[2l_{1,2p}]$ we have the equality $\Phi(\{.;l_{1,2p};l_{1,2p-1}\},\ldots) = 0$. If we put this result in RHS of (A.2) we get

$$\Phi(\{l_{1,2p+1};l_{1,2p};l_{1,2p-1}\},\ldots) = \Phi(\{l_{1,2p}+1;l_{1,2p};l_{1,2p-1}\},\ldots) = 1.$$

Thus, it is enough to prove statement (A.1) only for the case $l_{1,2p+1} = l_{1,2p}+1$.

In similar way we can prove that the statement (A.1) for arbitrary $l_{1,2p-1}$ is a consequence of statement for $l_{1,2p-1} = l_{1,2p}$ (this is superfluous if $l_{1,2p-1} = l_{1,2p}$ from the beginning). Thus, it is enough to prove the special case of statement (A.1) which is of the form

$$\Phi(\{l_{1,2p}+1;l_{1,2p};l_{1,2p}\},\ldots) = 1.$$

On the other hand,

$$\Phi(\{l_{1,2p}+1;l_{1,2p};l_{1,2p}\},\{l_{2,2p+1};l_{2,2p};l_{2,2p-1}\},\ldots) = \Phi(\{l_{2,2p+1};l_{2,2p};l_{2,2p-1}\},\ldots)$$

where all multipliers depending on $l_{1,2p+1};l_{1,2p};l_{1,2p-1}$ in the RHS are absent. This follows from the fact that

$$\phi^1(\{l_{1,2p}+1;l_{1,2p};l_{1,2p}\},\ldots) = 0,$$

$$\phi^1(\{l_{1,2p}+1;l_{1,2p}-1;l_{1,2p}\},\ldots) = 0,$$

$$\begin{aligned}&\phi^r(\{l_{1,2p}+1;l_{1,2p};l_{1,2p}\},\ldots)\\ &= \frac{\prod_{s=2}^{p} f(l_{s,2p+1};l_{r,2p})\prod_{s=1}^{p-1} f(l_{s,2p-1};l_{r,2p})}{\prod_{s\geq 2,s\neq r}^{p} f(l_{s,2p};l_{r,2p}) f(l_{s,2p}+1;l_{r,2p})}\,\frac{f(l_{1,2p}+1;l_{r,2p}) f(l_{1,2p};l_{r,2p})}{f(l_{1,2p};l_{r,2p}) f(l_{1,2p}+1;l_{r,2p})}\end{aligned}$$

in the LHS.

Thus, fulfillment of statement (A.1) for $p-1$ triples of variables implies validity of statement (A.1) for p triples . By repeating these arguments the analysis is reduced to the relation with single 'truncated' triple $\{l_{p,2p+1};l_{p,2p};.\}$ (corresponding to the $U_q(\mathrm{so}_3)$ case) :

$$\Phi(\{l_{p,2p+1};l_{p,2p};.\}) \equiv \frac{1}{[2l_{p,2p}]}(-f(l_{p,2p+1};l_{p,2p}) + f(l_{p,2p+1};l_{p,2p}-1)) = 1$$

whose validity is verified directly . □

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